

Submodular Function Maximization via the Multilinear Relaxation and Contention Resolution Schemes

Rico Zenklusen

MIT

Joint work with Chandra Chekuri and Jan Vondrák

Submodular functions

- ▶ Let N be a finite ground set, $n := |N|$.

Definition (submodular function)

A set function $f : 2^N \rightarrow \mathbb{R}$ is **submodular** if it has **diminishing returns**:

$$f(\underbrace{A+i}_{:=A \cup \{i\}}) - f(A) \geq f(B+i) - f(B) \quad \forall A \subseteq B \subseteq N, \forall i \in N \setminus B$$

Equivalent definition: $f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad \forall A, B \subseteq N.$

→ Submodularity is a natural property of utility functions.

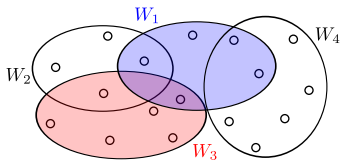
- ▶ f is **monotone** $\Leftrightarrow f(A) \leq f(B) \quad \forall A \subseteq B.$

Examples of submodular functions

Example I: coverage function

Let U be a finite ground set and $W_i \subseteq U$ for $i \in N$.

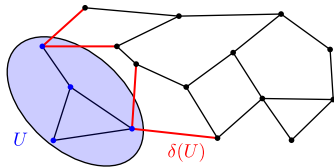
$$f(A) = \left| \bigcup_{i \in A} W_i \right| \quad \forall A \subseteq N$$



Example II: cut function

Let $G = (V, E)$ be a graph with edge weights $w : E \rightarrow \mathbb{R}_+$.

$$f(U) = w(\delta(U)) = w(E(U, V \setminus U)) \quad \forall U \subseteq V$$



Other examples

- ▶ Entropy function $H : 2^N \rightarrow \mathbb{R}_+$ of random variables $\{X_i\}_{i \in N}$:

$$H(A) := H(\{X_i \mid i \in A\}) \quad \forall A \subseteq N.$$

- ▶ Reduction of connection costs in facility location problems.
- ▶ ...

Optimizing submodular functions

Access to f by **value oracle**: can query $f(A)$ for $A \subseteq N$.

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Minimization vs. maximization

- ▶ Unconstrained minimization of submodular functions can be done efficiently.
-
- ▶ Unconstrained maximization of submodular functions is hard:
 - **No > 0.5 -approx** without exponentially many calls to value oracle. (Feige et al., 2007)
 - Remains hard in many settings outside the value oracle model (MAX-CUT, MAX-K-COVER, ...).
 - Currently best approximation ratio: **0.41**. (Oveis Gharan and Vondrák, 2011)

$\Theta(1)$ -approximations often achievable **under additional packing constraints**.

Some previous results on SFM (subm. funct. max.)

- ▶ Assume $f : 2^N \rightarrow \mathbb{R}_+$ (otherwise: no hope for good approximations).

Approaches for SFM are based either on

- greedy approaches,
- combinatorial local search procedures (replacing elements), or
- relaxation and rounding techniques.

Constraint type	Linear max.	Monotone subm. max.	Subm. max.
$O(1)$ knapsacks	$1 - \epsilon$	$1 - 1/e - \epsilon^1$	$0.25 - \epsilon^1$
1 matroid	1	$1 - 1/e^2$	0.325^3
$k = O(1)$ matroids	$1/(k - 1 + \epsilon)^4$	$1/(k + \epsilon)^4$	$1/(k + 1 + \frac{1}{k-1} + \epsilon)^4$

¹Kulik et al. (2011)

²Calinescu et al. (2011)

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Issue with previous approaches

Typically heavily tailored to the underlying constraints:

- ▶ unclear how to deal with combined constraints,
- ▶ no clear plan how to tackle new constraints.

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Typically **heavily tailored** to the underlying **constraints**:

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Is there some **more versatile framework**?

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Our results

- ▶ We introduce a rather **general framework** based on relaxation-and-rounding that
 - allows for **combining constraints** , and
 - provides a **general recipe** for SFM with packing constraints.

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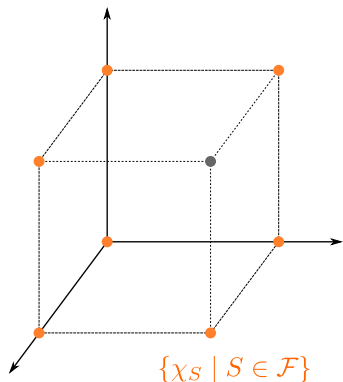
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(Some) new results due to our framework

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$k = O(1)$ matroids	$1/(k - 1 + \epsilon)$	$1/(k + \epsilon)$ for $k \geq 2$	$1/(k + 1 + \frac{1}{k-1} + \epsilon)$
k matr. & $\ell = O(1)$ knaps.	$0.6/k$	$0.38/k$	$0.19/k$
k -matchoid & ℓ -sparse PIP	$\Omega(1/(k + \ell))$	$\Omega(1/(k + \ell))$	$\Omega(1/(k + \ell))$
UFP on paths and trees	$\Omega(1)$	$\Omega(1)$	$\Omega(1)$

- new results
- previous results

General framework



1. Create relaxed problem

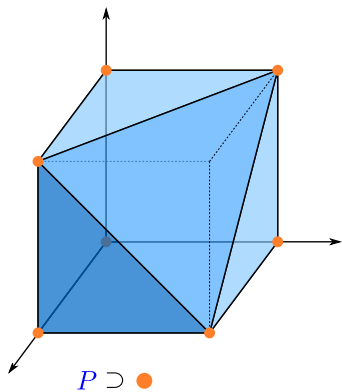
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 $\mathcal{F} \subseteq 2^N \rightsquigarrow$ polytope $P \subseteq [0, 1]^N$
- ii) Extend submodular function:
 $f \rightsquigarrow F : [0, 1]^N \rightarrow \mathbb{R}_+$

2. Maximize F over $P \rightsquigarrow x \in P$

3. Rounding: $x \rightsquigarrow I(x) \in \mathcal{F}$

- i) $x \rightsquigarrow R(x) \subseteq N$ with
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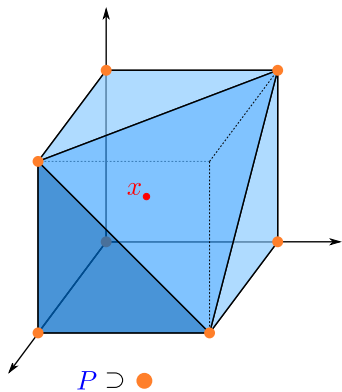
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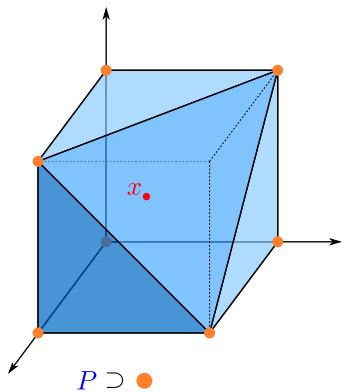
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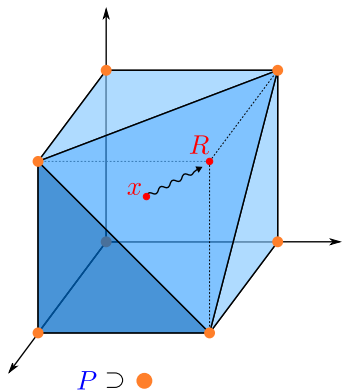
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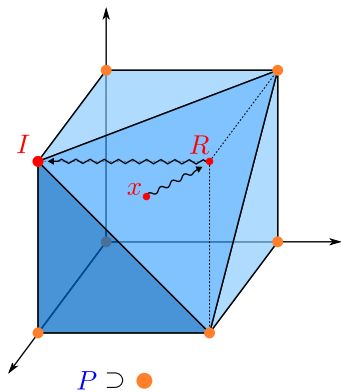
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The multilinear extension

Definition: multilinear extension F

$$F(x) := \sum_{S \subseteq N} f(S) \prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i) = E[f(R(x))], \text{ where}$$

$R(x) \subseteq N$: random set with $\Pr[i \in R(x)] = x_i$ independently for $i \in N$.

- ▶ Behaves nicely w.r.t. indep. rounding (would lead to constraint violations).
- ▶ Efficient approximate evaluation possible through Monte-Carlo sampling.

Improvements to the relax.-and-rounding framework

Maximization of F over a polytope P

- ▶ It is well understood how to maximize F for **monotone submodular** functions,
 - $(1 - \frac{1}{e})$ -approx for down-monot. & solvable P by Vondrák (2008).
 - ▶ For **non-monotone submodular** functions, not so much is known:
 - $(\frac{1}{4} - \epsilon)$ -approx for $O(1)$ knapsack constraints by Lee et al. (2009a),
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- ▶ Only little was known about how to round a point $x \in P$.
 - e.g. integrality gap of F over P unknown so far even if P is intersection of 2 matroids, and underlying submodular function is monotone.
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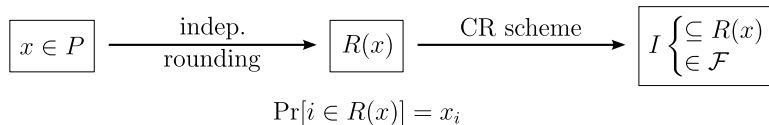
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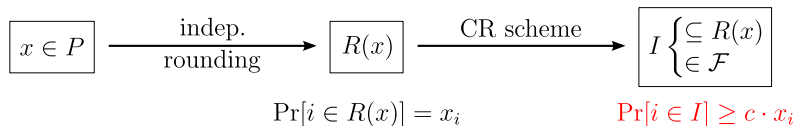
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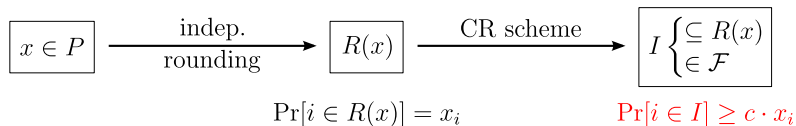
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Definition: balanced CR scheme

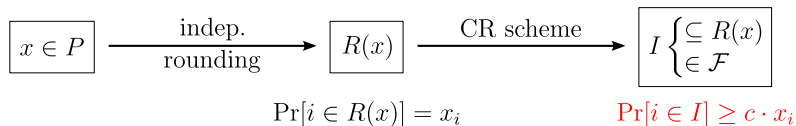
A **c-balanced CR scheme** for P is a (random) procedure parametrized by $x \in P$, that selects a set $I = I(x) \in \mathcal{F}$, $I \subseteq R(x)$ with

$$\Pr[i \in I] \geq c \cdot x_i \quad \Leftrightarrow \quad \Pr[i \in I \mid i \in R(x)] \geq c \quad \forall i \in N.$$

The scheme is called

- ▶ **monotone** if $\Pr[i \in I \mid R(x) = R_1] \geq \Pr[i \in I \mid R(x) = R_2] \quad \forall i \in R_1 \subseteq R_2$;

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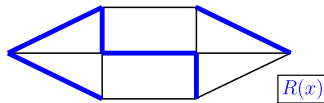
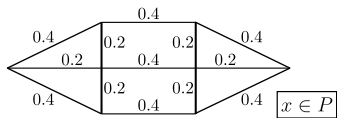
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Theorem (follows from Bansal et al. (2010))

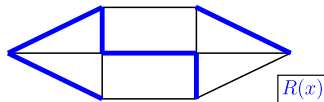
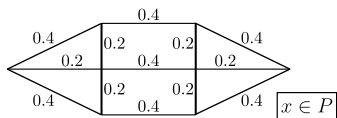
Let $x \in P$ and $I(x)$ be the output of a monotone c -balanced CR scheme (that satisfies $\Pr[i \in I \mid i \in R(x)] = c$). Then

$$\mathbf{E}[f(I(x))] \geq c \cdot F(x).$$

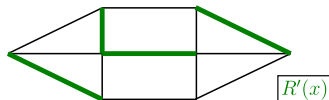
A simple $\frac{1}{8}$ -balanced CR scheme for matchings



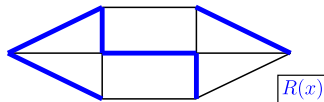
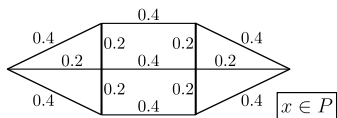
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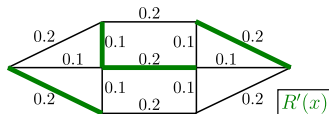
1. Remove each edge $e \in R(x)$ independently with probability $1/2 \rightarrow R'(x)$.



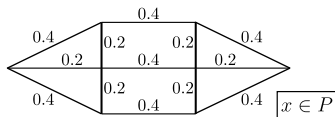
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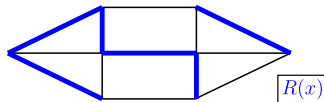
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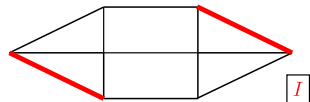
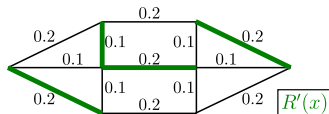
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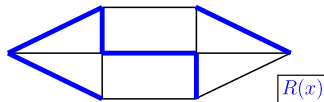
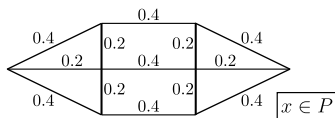
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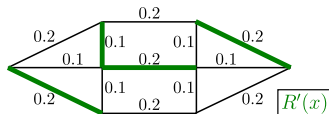
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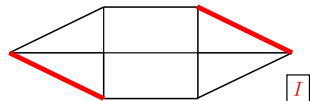
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- ▶ $\Pr[e \in I \mid e \in R'(x)] \geq \frac{1}{4}$,
since $e = \{u, v\}$ is only edge adjacent to u (or v) in $R'(x)$ with prob. $\geq 1/2$.
 $\Rightarrow \Pr[e \in I \mid e \in R(x)] = \underbrace{\Pr[e \in I \mid e \in R'(x)]}_{\geq 1/4} \underbrace{\Pr[e \in R'(x) \mid e \in R(x)]}_{=1/2} \geq \frac{1}{8}$.

- ▶ This CR scheme is indeed monotone.

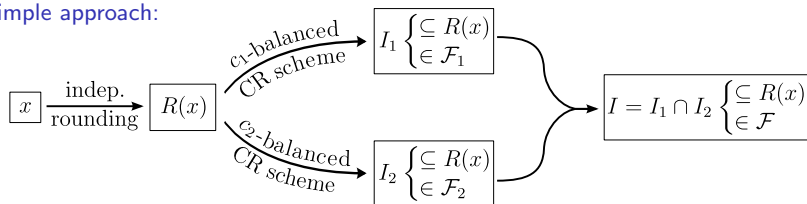
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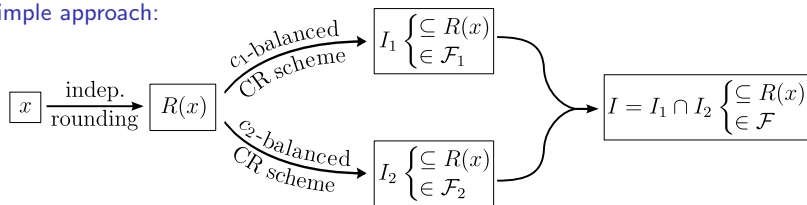
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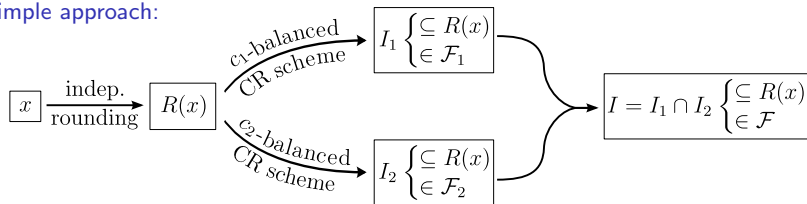


- ▶ Resulting CR scheme is $c_1 c_2$ -balanced (follows from FKG inequality).
- ▶ Monotonicity is preserved.

Combining CR schemes

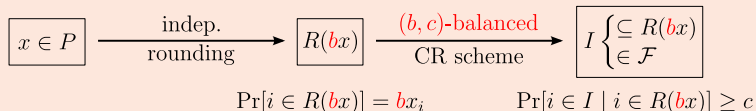
Often, \mathcal{F} is composed of simpler constraints: $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \Rightarrow P = P_1 \cap P_2$.

A simple approach:



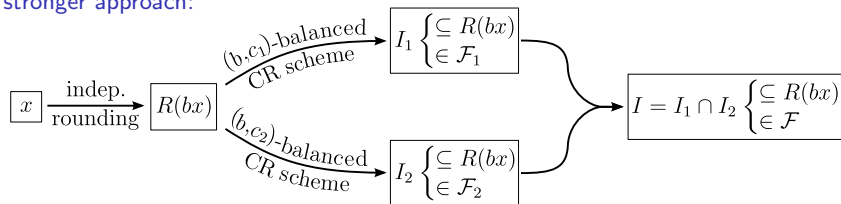
- ▶ Resulting CR scheme is $c_1 c_2$ -balanced (follows from FKG inequality).
- ▶ Monotonicity is preserved.

Often, **stronger combined schemes** can be obtained by first scaling down x a bit, $x \rightsquigarrow b \cdot x$, and **working with $R(bx)$ instead of $R(x)$** \rightarrow obtain higher values for c_1, c_2 .



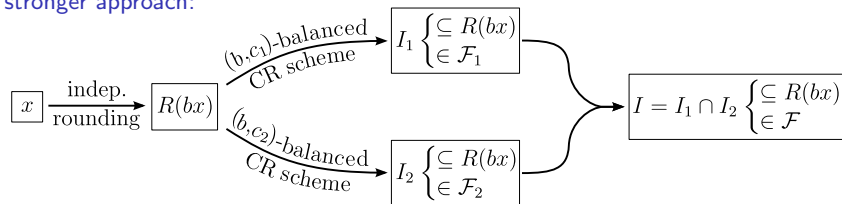
Combining CR schemes (II)

A stronger approach:



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A stronger approach:



- ▶ Resulting scheme is $b c_1 c_2$ -balanced.
- ▶ This approach is stronger in the parallel part.

Existence of strong CR scheme

Results on CR schemes

- ▶ $(b, \frac{1-e^{-b}}{b})$ -balanced, monotone and strict CR scheme for **matroid** constraint, for $b \in (0, 1]$. **This scheme is optimal.**
- ▶ For any fixed $\epsilon > 0$: $(1 - \epsilon, 1 - \epsilon)$ -balanced monot. and strict CR scheme for **knapsack** constraint.
- ▶ $(b, 1 - \Omega(b))$ -balanced, monotone and strict CR scheme for **UFP on trees**.
- ▶ $(b, 1 - 2kb)$ -balanced, monotone and strict CR scheme for **k -sparse PIP**.

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Putting the pieces together to obtain the claimed results

E.g. to optimize over k matroid constraints and a $\ell = \Omega(1)$ knapsacks, a c -balanced CR scheme can be obtained for

$$c = b \cdot \underbrace{\left(\frac{1 - e^{-b}}{b}\right)^k}_{\text{matroids}} \cdot \underbrace{(1 - \epsilon)^\ell}_{\text{knapsacks}} \stackrel{b=1/k}{=} \Omega(1/k).$$

$\Rightarrow \alpha \cdot \Omega(1/k) = \Omega(1/k)$ -**approx** to maximize f over those constraints, where $\alpha = 0.325$ is the approximation ratio for maximizing F over P .

Conclusions

- ▶ The **multilinear extension can be maximized** up to a constant factor on any **down-closed and solvable polytope**.
 - ▶ **Contention resolution schemes** provide a **modular** way for rounding a fractional point in the context of SFM.
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- ▶ What is the **best possible approximation ratio** for **maximizing F over P** ?
- ▶ Extend **techniques to find optimal/good CR scheme**?
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Thank you!

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