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A Simple Proof for a Characterization of Sign-Central Matrices using Linear Duality

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Outline

1. Problem definition and motivation
2. Complexity of deciding sign-centrality
3. Proof of the characterization using linear duality
4. A generalization to PSD
5. Conclusions

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Motivation

What can we say about the solutions of a class of linear systems if only the signs of the input matrices are known?

We consider a special case:

Under which conditions do we have that any matrix whose elements satisfy some fixed sign structure, has a non-negative non-zero element in its null-space?

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We consider a special case:

Under which conditions do we have that any matrix whose elements satisfy some fixed sign structure, has a non-negative non-zero element in its null-space?

Qualitative classes

Definition (Sign of a matrix)

With any matrix $A \in \mathbb{R}^{m \times n}$ we associate a sign matrix $sign(A) \in \{0, +, -\}^{m \times n}$:

$$\begin{pmatrix} 1 & -3 & 4 & 0 & -5 \\ 2 & 0 & -6 & 1 & -3 \\ 0 & -2 & 0 & 3 & 8 \end{pmatrix} \xrightarrow{sign} \begin{pmatrix} + & - & + & 0 & - \\ + & 0 & - & + & - \\ 0 & - & 0 & + & + \end{pmatrix} .$$

Definition (Qualitative class)

The qualitative class $Q(A)$ of a matrix $A \in \mathbb{R}^{m \times n}$ are the set of all matrices with the same sign as A, i.e.,

$$Q(A) := \{B \in \mathbb{R}^{m \times n} \mid sign(B) = sign(A)\} .$$

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$$Q(A) := \{B \in \mathbb{R}^{m \times n} \mid sign(B) = sign(A)\} .$$

Central and sign-central matrices

Definition (Central matrix)

$A \in \mathbb{R}^{m \times n}$ is central if there exists a nonnegative, non-zero element in its null-space, i.e., $\{x \in \mathbb{R}^n \mid Ax = 0, x \geq 0, x \neq 0\} \neq \emptyset$.

Definition (Sign-central matrix)

$A \in \mathbb{R}^{m \times n}$ is sign-central if every matrix in $Q(A)$ is central.

How to characterize sign-central matrices?

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How to characterize sign-central matrices?

Closed matrices

Definition (Closed matrix)

$A \in \mathbb{R}^{m \times n}$ is closed if for any non-strict sign vector $v \in \{\oplus, \ominus\}^m$, there is a column in A whose sign corresponds to v, i.e., $\exists\, j \in \{1, \ldots, n\}$ with

$$sign(A)_{ij} = \begin{cases} 0 \text{ or } + & \text{if } v_i = \oplus \\ 0 \text{ or } - & \text{if } v_i = \ominus \end{cases} \quad \forall i \in \{1, \ldots, m\} \ .$$

Closed matrices: Example

$$A = \begin{pmatrix} 1 & -3 & 4 & 0 & -5 \\ 2 & 0 & -6 & 1 & -3 \\ 0 & -2 & 0 & 3 & 8 \end{pmatrix}, \quad sign(A) = \begin{pmatrix} + & - & + & 0 & - \\ + & 0 & - & + & - \\ 0 & - & 0 & + & + \end{pmatrix}$$

The matrix A is closed since we find for all of the vectors

$$\begin{pmatrix} \oplus \\ \oplus \\ \oplus \end{pmatrix}, \begin{pmatrix} \oplus \\ \oplus \\ \ominus \end{pmatrix} \cdots \begin{pmatrix} \ominus \\ \ominus \\ \ominus \end{pmatrix}$$

a corresponding column in $sign(A)$.

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A characterization of sign-central matrices

Theorem ([Davidov and Davidova, 1992])

Let $A \in \mathbb{R}^{m \times n}$.

$$A \text{ is sign-central} \Leftrightarrow A \text{ is closed} \ .$$

- This theorem was first proven in [Davidov and Davidova, 1992] with a long and rather complicated proof.
- In [Ando and Brualdi, 1994] an alternative proof was presented using induction and a separation theorem of convex sets.
- Both proofs are hard to adapt to similar problems.

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We present a simple, short proof based on LP duality.

Why to bother about a new proof?

- The proof shows how duality theory can be used in the context of sign-systems $\longrightarrow$ well-established theory of convex duality can be used to tackle problems dealing with sign-systems.
- The approach used in the proof can be adapted for proving other characterization theorems (we will show an extension to PSD).
- The presented proof is very short and allows easily to get a good understanding of the characterization.

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Equivalence with tautologic expressions

With every matrix $A \in \mathbb{R}^{m\times n}$ we associate a Boolean expression $E(A)$ as shown below.

$$sign(A) = \begin{pmatrix} + & - & + & 0 & - \\ + & 0 & - & + & - \\ 0 & - & 0 & + & + \end{pmatrix} \quad \begin{matrix} \rightsquigarrow x_1 \\ \rightsquigarrow x_2 \\ \rightsquigarrow x_3 \end{matrix}$$

true:	$\oplus$
false:	$\ominus$

$$E(A) = (x_1 \wedge x_2) \vee (\overline{x_1} \wedge \overline{x_3}) \vee (x_1 \wedge \overline{x_2}) \vee (x_2 \wedge x_3) \vee (\overline{x_1} \wedge \overline{x_2} \wedge x_3)$$

Property

A is sign-central $\Leftrightarrow$ $E(A)$ is a tautology .

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Link with SAT problem

Property

A is not sign-central $\Leftrightarrow \neg E(A)$ allows a satisfying truth assignment .

Since $\neg E(A)$ is a SAT-problem in its general form we have:

Theorem

$\rightarrow$ *Deciding whether a matrix is sign-central is co-NP–complete.*

Remark

However, many interesting special cases are easy to solve, e.g., when the given matrix has a bounded number of zero entries per column.

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Preliminaries

Let $A \in \mathbb{R}^{m \times n}$. We define the following dual pair of LPs:

$$LP(A) \quad \boxed{\begin{array}{lrcl} \max & e^T x & & \\ s.t. & Ax & = & 0 \\ & x & \geq & 0 \end{array}} \qquad DP(A) \quad \boxed{\begin{array}{lrcl} \min & 0 & & \\ s.t. & A^T y & \geq & e \\ & y & \text{free} & \end{array}},$$

where $e = (1, \dots, 1) \in \mathbb{R}^n$.

Property 1

A is central $\Leftrightarrow OPT(LP(A)) = \infty \Leftrightarrow DP(A)$ is infeasible .

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A sign-central $\Rightarrow$ A closed (1)

Suppose A not closed $\Rightarrow \exists\, v \in \{\oplus, \ominus\}^m$ s.t. no column of A corresponds to v.

We will define a matrix B and a vector $y \in \mathbb{R}^m$ satisfying $sign(B) = sign(A)$ and $B^T y \geq e$ ($\rightarrow$ $DP(B)$ is feasible).

We choose

$$y_i = \begin{cases} 1 & \text{if } v_i = \ominus \\ -1 & \text{if } v_i = \oplus \end{cases} \quad \forall i \in \{1, \ldots, m\} \ .$$

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A sign-central $\Rightarrow$ A closed (2)

We start with $B = A$ and modify B to satisfy $B^T y \geq e$. Let $j \in \{1, \ldots, n\}$ and consider $(B^T y)_j$. Since B has no column whose sign structure is opposite to y, we have

$$\exists\, i \in \{1, \ldots, m\} \text{ with } B_{ij} \cdot y_i > 0 \ .$$

By choosing $|B_{ij}|$ large enough we ensure $(B^T y)_j \geq 1$. By repeating the process for every $j \in \{1, \ldots, n\}$ we finally get

$$B^T y \geq e \ .$$

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A sign-central $\Leftarrow$ A closed

By contradiction we assume that A is not sign-central but closed.

$$\xrightarrow{A \text{ not sign-central}} \quad \exists\, B \in \mathbb{R}^{n\times m} \text{ not central with } sign(B) = sign(A).$$

$$\xrightarrow{\text{by Property 1}} \quad \exists\, y \in \mathbb{R}^{m} \text{ with } B^T y \geq e.$$

Since B is closed, B contains a column whose sign structure is opposite to y, i.e.,

$$\exists\, j \in \{1, \ldots, n\} \text{ with } (B^T y)_j \leq 0 \ .$$

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PSD sign-centrality

Definition (PSD central matrix familiy)

We call a family of matrices $\{A^k\}_{k\in\{1,\dots,r\}} \subset \mathbb{R}^{n\times n}$ *PSD central if*

$$\{X \succeq 0 \mid A^k \bullet X = 0 \ \ \forall k \in \{1,\dots,r\}, X \neq 0\} \neq \emptyset \ \ .$$

Definition (PSD sign-central matrix family)

A family $\{A^k\}_{k\in\{1,\dots,r\}} \subset \mathbb{R}^{n\times n}$ *is called PSD sign-central if every family* $\{B^k\}_{k\in\{1,\dots,r\}} \subset \mathbb{R}^{n\times n}$ *with* $sign(B_k) = sign(A_k) \forall k$ *is central.*

Remark

The above definition for sign-centrality of PSD is a generalization of classical sign-centrality:
→ Choose all A^k to be diagonal matrices.

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A characterization for PSD sign-centrality

Again we can define an appropriate pair of primal and dual problems:

$LP(\{A^k\})$

$$\begin{array}{llll} \max & I \bullet X & & \\ s.t. & A^k \bullet X = 0 & \forall\, k \\ & X \succeq 0 & \end{array}$$

$DP(\{A^k\})$

$$\begin{array}{lll} \min & 0 & \\ s.t. & \sum_{k=1}^{r} \lambda_k A^k \succeq I & \\ & \lambda_k \; free \; \forall\, k & \end{array}$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

Theorem

$$\{A^k\} \text{ is PSD sign-central} \Leftrightarrow \begin{pmatrix} diag(A^1)^T \\ \vdots \\ diag(A^r)^T \end{pmatrix} \text{ is closed} .$$

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Conclusions

- New approach for proving characterizations of sign-systems:
  - Based on duality theory.
  - Simple, short proof of a classical characterization.
  - Underlines that the characterization is essentially a theorem of alternatives.
- The approach can be adapted to profit from convex duality theory for proving more general characterizations – Example: PSD sign-centrality.
- We hope that the shown link to duality theory allows to find and prove further results for sign-systems.

References I

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G. Davidov and I. Davidova. Tautologies and positive solvability of linear homogeneous systems. *Annals of Pure and Applied Logic*, 1992.