

A bound on the collection of edges in MST's of fixed-size subgraphs

Rico Zenklusen

Institut for Operations Research, D-MATH, ETH Zurich

`rico.zenklusen@ifor.math.ethz.ch`

Joint work with Gregory B. Sorkin and Angelika Steger

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Outline

1 Introduction

- The conjecture
- Interpretation using k -constructible graphs
- Related work

2 Proof of the conjecture

- Inductive approach
- Splitting into G^1 and G^2

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Conjecture of Goemans and Vondrák

Definition

Let $G = (V, E, w)$ be a complete n -vertex graph with distinct positive edge weights w . For $k \in \{1, 2, \dots, n-1\}$ We define

$$M_k(G) = \bigcup_{X \subseteq V, |X|=k-1} \text{MST}(G \setminus X).$$

Theorem (Conjecture of Goemans and Vondrák)

$$|M_k(G)| \leq nk - \binom{k+1}{2}$$

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Menger's theorem

In further discussions we regularly use a special form of Menger's theorem.

Definition

Two vertices s, t in an undirected graph G are called k -connected if there exist k vertex disjoint paths connecting them in G .

Theorem

Let s, t be two non-adjacent vertices in an undirected graph G . s and t are k -connected in G if and only if after deleting $k - 1$ vertices (distinct from s and t) s and t are still connected.

Which edges are in $M_k(G)$?

Some notations

- $E = \{e_1, e_2, \dots, e_m\}$ with $w(e_1) < w(e_2) < \dots < w(e_m)$.
- $G_i = (V, \{e_1, e_2, \dots, e_i\})$.

Theorem

An edge $e_i \in E$ is exactly then in $M_k(G)$ when G_{i-1} contains at most $k - 1$ vertex disjoint paths between the two endpoints of e_i .

→ direct consequence of Menger's theorem

k -constructible graphs

Definition

A graph $G = (V, E)$ is called **k -constructible** if there exist an ordering $O = (e_1, e_2, \dots, e_{|E|})$ of the edges in E such that for all $i \in \{1, 2, \dots, m\}$ the graph $(V, \{e_1, e_2, \dots, e_i\})$ contains at most k vertex disjoint paths between the two endpoints of e_i . The order O is called a **k -construction order** for G . The triple (V, E, O) is called a **k -construction**.

- $\mathcal{G}_{n,k} :=$ set of all k -constructible graphs with n vertices

k -constructible graphs (continued)

Theorem

For every complete graph $G = (V, E, w)$ with n vertices and distinct positive edge weights w we have

$$M_k(G) \in \mathcal{G}_{n,k} \quad \forall k \in \{1, 2, \dots, n-1\}.$$

We will prove the following theorem implying the conjecture of Goemans and Vondrák.

Theorem

For $G = (V, E) \in \mathcal{G}_{n,k}$ we have

$$|E| \leq nk - \binom{k+1}{2}.$$

k -constructible graphs (continued)

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k -minimal graphs

Definition

A graph $G = (V, E)$ is a k -minimal graph if it is k -connected and removing any edge of G destroys k -connectivity.

Property

Every k -minimal graph is k -constructible (any order of its edges is a k -construction order) *but not the converse*.

Theorem (Mader 1971)

Every k -minimal graph $G = (V, E)$ with n vertices satisfies

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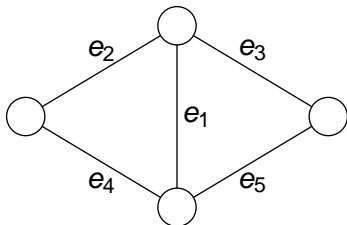
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Every k -minimal graph $G = (V, E)$ with n vertices satisfies

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Example: k -constructible graph not being k -minimal

The graph below is 2-constructible with 2-construction order $(e_1, e_2, e_3, e_4, e_5)$ but eliminating the edge e_1 does not destroy 2-connectivity.



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Inductive hypothesis and base step

We fix $k \in \mathbb{N}$.

Inductive hypothesis

The conjecture is true for k -constructible graphs with number of vertices $\in \{k + 1, k + 2, \dots, n - 1\}$.

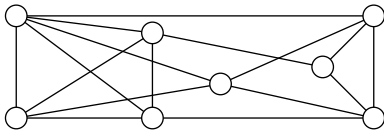
Base step

For graphs $G = (V, E)$ with $k + 1$ vertices, the conjecture states

$$|E| \leq \binom{k + 1}{2}$$

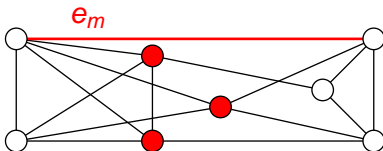
and is thus trivially true.

Splitting into two k -constructible graphs

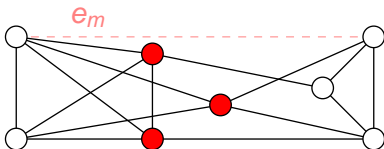


$G = (V, E)$, a 4-constructible graph

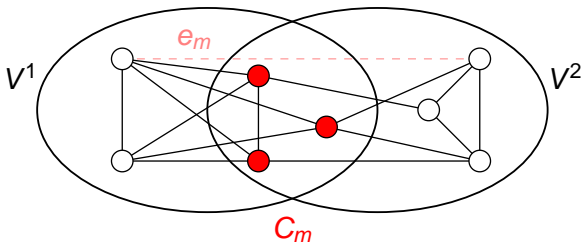
Splitting into two k -constructible graphs



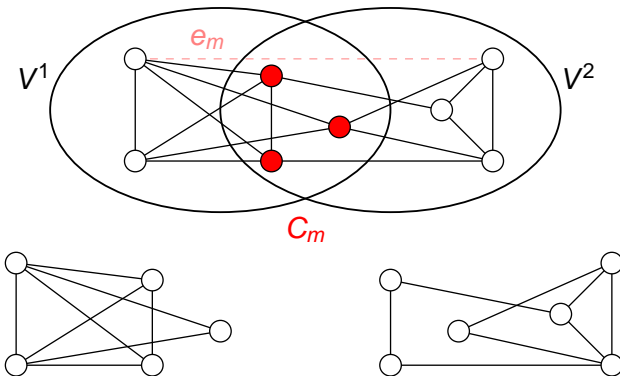
Splitting into two k -constructible graphs



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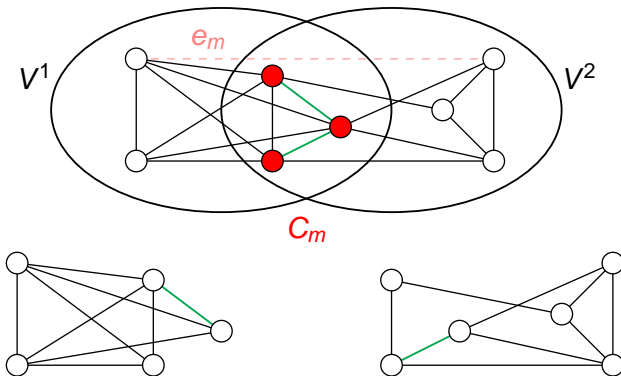


Splitting into two k -constructible graphs



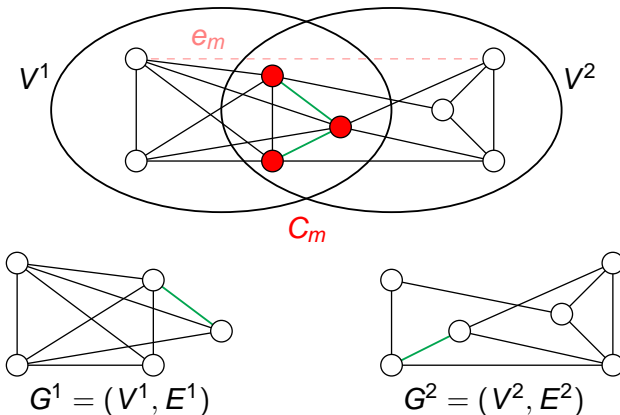
Splitting into two k -constructible graphs

$$B = (C_m \times C_m) \setminus E$$



Splitting into two k -constructible graphs

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Applying the inductive hypothesis

$$|E^1| + |E^2| = \underbrace{(|E| - 1)}_{\text{I}} + \underbrace{|G[C_m]|}_{\text{II}} + \underbrace{\left(\binom{k-1}{2} - |G[C_m]| \right)}_{\text{III}}$$

The terms in the above sum represent:

- I**: the number of edges in G without e_m
- II**: the double-counting of the edges $G[C_m]$
- III**: the number of added edges in $C_m \times C_m$

Applying the inductive hypothesis

$$\begin{aligned} |E^1| + |E^2| &= (|E| - 1) + |G[C_m]| + \left(\binom{k-1}{2} - |G[C_m]| \right) \\ \Rightarrow |E| &= 1 + |E^1| + |E^2| - \binom{k-1}{2} \end{aligned}$$

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Applying the inductive hypothesis on G^1 and G^2 , we finally get

$$\begin{aligned} |E| &\leq 1 + \left(|V^1|k - \binom{k+1}{2} \right) + \left(|V^2|k - \binom{k+1}{2} \right) - \binom{k-1}{2} \\ &= nk - \binom{k+1}{2}. \end{aligned}$$

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Choice of G

It suffices to show the correctness of the conjecture for $G = (V, E) \in \mathcal{G}_{n,k}$ having the maximum number of edges among all graphs in $\mathcal{G}_{n,k}$.

Notations

- $O = (e_1, e_2, \dots, e_m)$: a k -construction order for G .
- $G_i = (V, \{e_1, e_2, \dots, e_i\})$.
- $r_1^j < r_2^j < \dots < r_{m_j}^j$: indices of the edges in E having both endpoints in $V^j \quad \forall j \in \{1, 2\}$

In the next steps we will build two k -construction orders O^1, O^2 corresponding to G^1 and G^2 . The order O^j be built by starting with $(e_{r_1^j}, e_{r_2^j}, \dots, e_{r_{m_j}^j})$ and adding edges of B into this order.

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Partitioning B

We partition $B (= (C_m \times C_m) \setminus E)$ into the following sets B_1, B_2, \dots, B_m .

$$B_i = \left\{ \{v_1, v_2\} \in B \mid \begin{array}{l} v_1 \text{ and } v_2 \text{ are } k\text{-connected in } G_i \text{ and} \\ k-1\text{-connected in } G_{i-1} \end{array} \right\}$$

Definition

$$\tilde{G}_i := (V, \{e_1, e_2, \dots, e_i\} \cup \bigcup_{l=1}^i B_l)$$

Theorem (A)

For all $i \in \{1, 2, \dots, m\}$ we have that the two endpoints of e_i are at most $k-1$ -connected in \tilde{G}_{i-1} .

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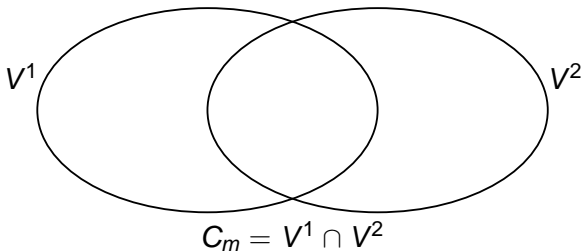
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Proof of theorem (A)

- Let $C_i \subseteq V$ with $|C_i| = k - 1$ be a set separating the two endpoints of e_i in G_{i-1} .
- $\forall e \in \bigcup_{l=1}^i B_l$, C_i cannot separate the two endpoints of e in G_{i-1} because they are at least k -connected.
- Thus the edges in $\bigcup_{l=1}^i B_l$ only connect vertices in $G_{i-1}[V \setminus C_i]$ which are already in the same connected component.
- C_i is a separating set for the two endpoints of e_i in \tilde{G}_i .

Structure of O^j

$$O = (e_1, e_2, \dots, e_m)$$

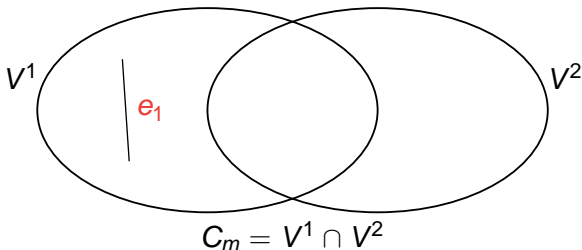


Constructing O^1
()

Constructing O^2
()

Structure of O^j

$$O = (e_1, e_2, \dots, e_m)$$



→ Partition of B_1 into B_1^1, B_1^2

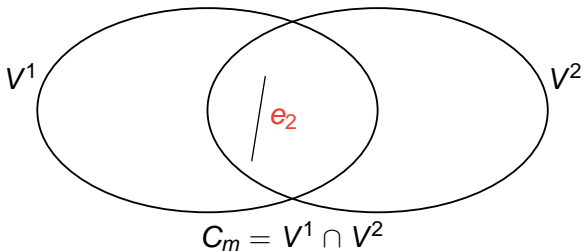
→ Define order O_1^j for elements in B_1^j

Constructing O^1
(e_1, O_1^1)

Constructing O^2
(O_1^2)

Structure of O^j

$$O = (e_1, e_2, \dots, e_m)$$



→ Partition of B_2 into B_2^1, B_2^2

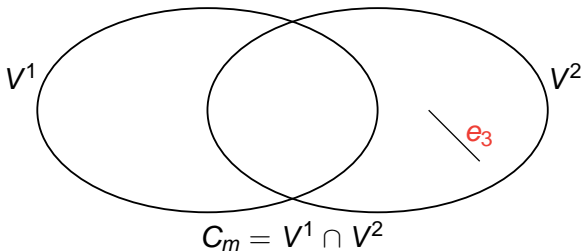
→ Define order O_2^j for elements in B_2^j

Constructing O^1
(e_1, O_1^1, e_2, O_2^1)

Constructing O^2
(O_1^2, e_2, O_2^2)

Structure of O^j

$$O = (e_1, e_2, \dots, e_m)$$



→ Partition of B_3 into B_3^1, B_3^2

→ Define order O_3^j for elements in B_3^j

Constructing O^1
($e_1, O_1^1, e_2, O_2^1, O_3^1$)

Constructing O^2
($O_1^2, e_2, O_2^2, e_3, O_3^2$)

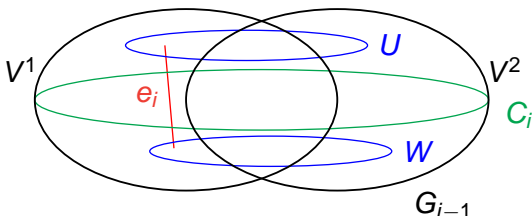
Structure of B_i

We fix $i \in \{1, 2, \dots, m\}$ and define

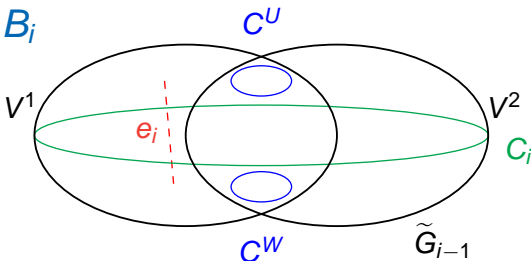
- $C_i \subset V$ with $|C_i| = k - 1$: a set separating the two endpoints of e_i in G_{i-1} .
- $U, W \subseteq V \setminus C_i$: the two components in $G_{i-1}[V \setminus C_i]$ containing the two endpoints of e_i
- $C^U = C_m \cap U$, $C_m^W = C \cap W$

Property

$$B_i \subseteq C^U \times C^W$$

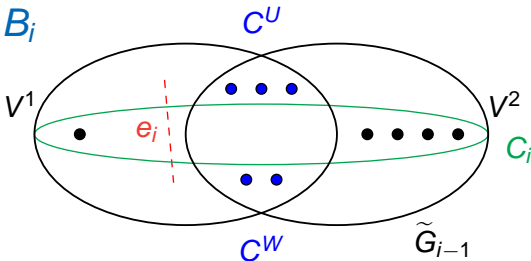


Splitting B_i



- $S^j = C_i \cap V^j$ separates the two endpoints of e_i in $\tilde{G}_{i-1}[V^j]$
- We can add any $k - 1 - |S^j|$ edges to the current graph j without violating k -constructibility
- $\max_{j \in \{1,2\}} \{k - 1 - |S^j|\} \geq \min\{|C^U|, |C^W|\}$

Splitting B_i



- Suppose $|C^U| \geq |C^W|$, and $|V^1| \leq |V^2|$. Choose $v \in C^U$.
- Add all edges of B_i being adjacent to v in any order to O_i^1
- $S^1 \cup \{v\}$ separates all remaining edges in B_i in the current graph over V^1 and S^2 still separates all remaining edges of B_i in the graph over V^2 .
- Repeat the previous steps over the remaining edges in B_i until all edges are associated to one of the two orders O_i^1, O_i^2 .